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RECURSIVE ENUMERABILITY AND INDEXICALITY IN E-RECURSION

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§ 0 Introduction

The question of the limits of recursive enumerability and indexicality were first formulated by Sacks (see Sacks [1980] or Sacks-Griffor [1980]). E-recursion or 'set recursion' as a natural generalization of Kleene recursion in normal objects of finite type was introduced by Normann [1978] and rediscovered independently by Moschovakis [1976].

If we consider E-closed ordinals, Sacks [1980 b] showed that if $L_\varkappa = E(\gamma)$ for some $\gamma < \varkappa$ and $L_\varkappa$ is not Σ_1- admissible such that:

$$L_\varkappa \models \text{"gc}(\varkappa) \text{ is regular"}, \qquad (\text{gc}(\varkappa)$$

is the greatest cardinal in the sense of $L_\varkappa$ and exists by the assumption of inadmissibility), then $L_\varkappa$ is not $\underset{\sim}{RE}$, i.e. there is no procedure with parameter in $L_\varkappa$ which is defined only on elements of $L_\varkappa$. Sacks also indicated a proof that if

$$L_\varkappa \models \omega < cf(gc(\varkappa)) < gc(\varkappa)\text{, then}$$

$\mathcal{P}(gc(\varkappa)) \cap L_\varkappa$ is $\underset{\sim}{RE}$.

In this paper we consider arbitrary E-closed $L_\varkappa$. In the case of inadmissible E-closures such that:

$$L_\varkappa \models \omega < cf(gc(\varkappa)) < gc(\varkappa)$$

we develope the method of Sacks to not only show that $L_\varkappa$ is $\underset{\sim}{RE}$, but further that $L_\varkappa$ is indexical and REC! In the case of admissible E-closures (i.e. HYP(γ) for some $\gamma \in OR$) we have that $L_\varkappa$ is $\underset{\sim}{RE} = \Sigma_1$. Thus when $L_\varkappa$ is $\underset{\sim}{RE}$ is completely answered for E-closed $L_\varkappa$'s which are E-closures.

Finally, if $L_\varkappa$ is E-closed but not the E-closure of one of its elements, we almost completely determine when $L_\varkappa$ is $\underset{\sim}{RE}$ for countable $L_\varkappa$. Remarks follow these results indicating where the corresponding methods can be used in the uncountable case. For details on forcing in E-recursion see Griffor [1982].

§ 1 Some Background.

An effective enumeration of the universe for computation is something one might well expect. As Sacks showed this is not always the case despite the fact that initial segments of L can be 'enumerated' in order of constructibility. Let $L_\varkappa$ be E-closed.

Definition 1.1 (i) $L_\varkappa$ is $\underset{\sim}{RE}$, if $\exists a \in L_\varkappa\ \exists e \in \omega$ such that for all $x \in V$

$$x \in L_\varkappa \iff \{e\}(a,x)\downarrow ;$$

(ii) if $L_\varkappa = E(\gamma)$ for some $\gamma < \varkappa$, then $L_\varkappa$ is indexical, if $\exists a \in L_\varkappa$ such that for all $x \in L_\varkappa$ there is an $I_x \subseteq \gamma$ such that

(a) $I_x \neq \emptyset$ and $I_x \leq_E x, a$ and

(b) $(\forall \delta \in I_\gamma)[x \leq_E \delta, a]$

Remark Initially Sacks defined a set $R \subseteq 2^\omega$ such that $R \leq {}^3\mathbb{E}, a$ for some $a \in 2^\omega$ to be indexical, if $\exists I \subseteq 2^\omega$ such that

(i) $I \neq \emptyset$ and $I \leq {}^3\mathbb{E}, R$ and

(ii) $(\forall b \in I)[R \leq {}^3\mathbb{E}, b]$, where

the structure in question was the companion to Kleene recursion in ${}^3\mathbb{E}$. It was under the assumption of a recursive well-ordering of 2^ω which is recursively regular that Sacks produced non-indexical $R \in 2\text{-}\underset{\sim}{sc}({}^3\mathbb{E})$ in showing that $2 - \underset{\sim}{sc}({}^3\mathbb{E})$ was not $\underset{\sim}{RE}$.

In the setting of the ordinals, (i.e. E-closed L_κ such that $L_\kappa = E(\gamma)$ for some $\gamma < \kappa$), indexicality amounts to every set being recursively equivalent to an ordinal modulo a parameter.

The intuition here is that every set which is computed is computed at a level, namely, its order of computability. In L this corresponds to the sets order of constructibility. A universe for computation is 'indexical' if, relative to a parameter in that universe, we can pass effectively from a set to its order of computability.

Proposition 1.2. Consider $L_{\kappa_1^{3\mathbb{E}}}(2^\omega)$, then if $\alpha < \kappa_1^{3\mathbb{E}}$ we have α is indexical.

proof using 2^ω consider

$I_\alpha = \{b \in 2^\omega \mid b$ codes a convergent computation of length $\alpha\}$

then $I_\alpha \leq {}^3\mathbb{E}, \alpha$: if $b \in I_\alpha$ we can recognize it using b, otherwise it has length less than α or is not a code for a computation or has a subcomputation of length α.

Remark. If there is a recursive well-ordering of reals, then if x is indexical, we can take I_x to be a singleton by choosing the least such.

If $L_\varkappa = E(\gamma)$ for some $\gamma < \varkappa$ and γ is regular in $L_\varkappa$, then Sacks concluded that non-indexical sets existed by the following implicit lemma.

<u>Lemma 1.3.</u> If $L_\varkappa = E(\gamma)$ for some $\gamma < \varkappa$, then

$$L_\varkappa \text{ indexical} \Rightarrow L_\varkappa \text{ is } \underset{\sim}{RE}$$

<u>proof</u> let $a \in L_\varkappa$ witness indexicality and for $x \in V$ compute, via a, $I_x \subseteq \gamma$ such that its elements compute x, otherwise diverge.

<u>Remark</u> If $x \in V$ is transitive, then it remains open whether $E(x)$ being $\underset{\sim}{RE}$ is equivalent with $E(x)$ being indexical. Proofs that inadmissible $E(\gamma)$'s are $\underset{\sim}{RE}$ proceed by showing that $E(\gamma)$ is indexical. An essential difference between the two properties is that indexicality is internal while being $\underset{\sim}{RE}$ makes reference to $x \in V/L_\varkappa$. Hence, for example, absoluteness considerations apply to the first, but not the second.

We shall be concerned primarily with E-closed initial segments of L. As remarked before, order of computability is the key to indexicality and, in L, reduces to order of constructibility.

<u>Definition 1.4.</u> Let $x \in V$ be transitive and for $y \in E(x)$, let

$$\begin{aligned} O^x(y) &= \mu\gamma < OR \cap E(x)[y \text{ is computed} \\ &\quad \text{from } x \text{ and some } b \in x \text{ via a} \\ &\quad \text{computation of length } \gamma] \\ &= \text{'order of computability' of } x. \end{aligned}$$

<u>Remark</u>. With $E(x)$ as above we have that

$$E(x) \text{ is indexical} \iff$$

$$(\exists y \in E(x))(\forall z \in E(x))[\varkappa_r^{y,z} > 0^x(z)].$$

This follows immediately from the fact that $\varkappa_r^{y,z}$ is the greatest y,z-reflecting ordinal.

§ 2 E-closures.

We begin our analysis of which E-closed $L_\varkappa$'s are $\underset{\sim}{RE}$ with the case of $L_\varkappa = E(\alpha')$ not Σ_1-admissible. Then $L_\varkappa$ has a greatest cardinal (unbounded cardinals would yield admissibility) written $\alpha = gc(\varkappa)$ and we let $\gamma = cf^{L_\varkappa}(\alpha)$. By Kirousis [1980]. $\gamma > \omega$, for otherwise $L_\varkappa$ is admissible. If $\gamma = \alpha$ Sacks' argument showed that $L_\varkappa$ is not $\underset{\sim}{RE}$. If $\gamma < \alpha$ one might expect that being $\underset{\sim}{RE}$ had something to do with admissibility and, hence, that $L_\varkappa$ is not $\underset{\sim}{RE}$.

<u>Theorem 2.1</u>. Let $L_\varkappa = E(\alpha')$ be inadmissible with $\alpha = gc(\varkappa)$ and $\gamma = cf^{L_\varkappa}(\alpha)$, then

$$\gamma < \alpha \Rightarrow L_\varkappa \text{ is } \underset{\sim}{REC}$$

(i.e. $L_\varkappa$ is $\underset{\sim}{RE}$ and $V/L_\varkappa$ is $\underset{\sim}{RE}$) both via parameters <u>in</u> $L_\varkappa$!

<u>proof</u> Notice that γ is a regular $L_\varkappa$-cardinal and work in $L_\varkappa$. Let $A \in L_\varkappa$. If $O(A) = \mu\gamma < \varkappa[A \in L_\gamma]$ then

$$0^{x'}(A) = O(A).$$

Assume that $A \subseteq \alpha$. S. Friedman [1980a] gave an analysis of the α-degrees of subsets of $\aleph_{\omega_1}$ where $\alpha = \aleph_{\omega_1}$ (α-degrees in the sense of α-recursion theory on $L_{\delta_{\omega_1}}$.) The same analysis is valid in $L_\varkappa$:

Let $h : \gamma \to \alpha$ witness the cofinality of α in $L_\varkappa$ and following Friedman define the 'cut-off function' of A:

$$f_A : \gamma \to \alpha \text{ by}$$

$$f_A(\delta) = \beta, \text{ if } A \cap h(\delta) \text{ is the } \beta\text{th element of } L.$$

Inside $L_\varkappa$ we can recursively decide which ordinals $<\alpha$ are $L_\varkappa$-cardinals and w.l.o.g. we may assume that $h(\delta)$ is an $L_\varkappa$-cardinal for $\delta < \gamma$, thus

$$f_A(\delta) < h(\delta)^+ \text{ (the next}$$

cardinal in $L_\varkappa$).

Lemma 2.2. (S. Friedman) Let $A, B \in L_\varkappa$, $A \subseteq \alpha$ and $B \subseteq \alpha$ and assume that

$$\{\delta < \gamma \mid f_A(\delta) \leq f_B(\delta)\}$$

is stationary. Then $A \leq_\alpha B, h$ (A is α-recursive in B, h).

proof (sketch) Let g_δ be a 1-1 map of $f_B(\delta) + 1$ onto $h(\delta)$ and let

$$t(\delta) = \delta', \text{ if } \delta' \text{ is minimal such that}$$

$$g_\delta(f_A(\delta)) < h(\delta').$$

If δ is a limit ordinal in H, then $t(\delta) < \delta$ and by Fodor's theorem (which is valid in $L_\varkappa$) we have that t is bounded on a stationary set $H_1 \subseteq \gamma$. Let δ_1 be the bound and then for some $\delta_2 < \delta_1$, t is constant δ_2 on some unbounded subset of γ.

Then relative to this unbounded set, δ_2, h and B we can compute A α-recursively.

Corollary 2.3.

(i) If $A,h <_\alpha B,h$, then $\{\delta<\gamma \mid f_A(\delta)<f_B(\delta)\}$ contains a closed unbounded subset of γ;

(ii) If $\alpha \leq O(A) < O(B)$, then $\{\delta<\gamma \mid f_A(\delta)<f_B(\delta)\}$ contains a closed unbounded subset of γ;

(iii) $<_\alpha$ (relative to h) is a prewellordering on $L_\varkappa \cap \mathcal{P}(\alpha)$.

proof (i) if not then $\{\delta<\gamma \mid f_B(\delta)\leq f_A(\delta)]$ is stationary and so $B,h \leq_\alpha A,h$.

(ii) If not, then $B,h \leq_\alpha A,h$ and so $O(B) \leq O(A)$.

(iii) $<_\alpha$ (set to h) is clearly a preorder. If a part of it is not wellfounded we would have a descending sequence (here a Moschovakis witness) in $L_\varkappa$ which would give a countable collection of club sets on γ in $L_\varkappa$ with empty intersection in $L_\varkappa$, contradicting

$L_\varkappa \models$ 'the club filter or γ is countably additive'.

(Note that we have used that $L_\varkappa$ is not Σ_1-admissible.)

Now the order of A in $<_\alpha$ (rel. to h) excedes $O(A)$ (literally, give or take α) and thus $O(A)$ is computable in A. So

$$A \in L_\varkappa \iff A \in L_{\alpha+\|A\|_{<_\alpha}(h)}$$

Thus $L_\varkappa \cap \mathcal{P}(\alpha)$ is $\underset{\sim}{RE}$ and by the previous remark we have in fact shown that $L_\varkappa \cap \mathcal{P}(\alpha)$ is indexical.

<u>Lemma 2.4.</u> Let $L_\varkappa$ be as above. If $\mathcal{P}(\alpha) \cap L_\varkappa$ is indexical (on $L_\varkappa$), then $L_\varkappa$ is indexical.

<u>proof</u> we proceed by transfinite recursion on the rank of $x \in L_\varkappa : \rho(x)$. By induction hypothesis we have succeeded in computing $O(z)$ from z for all $z \in x$. Compute

$$\sup_{z \in x} O(z) = \tau < \varkappa, \quad \text{then}$$

$x \subseteq L_\tau$. We can now pass effectively from τ to

$$f_\tau : \tau <-> \alpha \quad \text{and; using the}$$

identification between τ and L_τ, compute $f_\tau{}'' x \subseteq \alpha$. By assumption $f_\tau{}'' x$ is indexical and we can therefore compute a set of indices for x. By effective transfinite recursion we have given an algorithm uniformly in α, x and the recursion theorem gives the desired parameter witnessing the indexicality of $L_\varkappa$.

Thus $L_\varkappa$ is indexical and by a previous lemma $L_\varkappa$ is $\underset{\sim}{\mathrm{RE}}$. We will now show that if $A \notin L_\varkappa$, then this too can be verified by a computation. The intuition is that if $A \notin L_\varkappa$ then we use the club filter on γ to compute the order of A in $<_\alpha(h)$ which must be $\geq_\varkappa$ and hence we can compute $\varkappa$ from A, α. If $\omega_1^{L_\varkappa} = \omega_1^L$ then this line of reasoning will succeed, but $\omega_1^{L_\varkappa}$ may be less than ω_1^L.

Let $A \subseteq \alpha$ and let B_β be the βth subset of α α-recursive in A, h. If one of the $f_\beta = f_{B_\beta}$'s fails to be definable over α, we know that $A \notin L_\varkappa$. In fact we may assume that:

(i) Each f_β can be defined;

(ii) $\beta_1 <_A \beta_2$ iff $\{\delta < \gamma \mid f_{\beta_1}(\varepsilon) < f_{\beta_2}(\delta)\}$

contains a closed unbounded subset of γ is a pre-linear order;

(iii) for each β_1, β_2 the set

$$\{\delta < \gamma \mid f_{\beta_1}(\delta) \leq f_{\beta_2}(\delta)\} \in L_\kappa$$

since $L_\kappa \models (\gamma)$ exists. We can code α' as a relation on α, so instead of $E(\alpha')$ we can consider recursion in a type-2 functional over α. The advantage is that all computations may be described as elements of α (one may as well use a notation system).

The set of computations and the comparison of lengths upon them is defined by a positive inductive definition Γ s.t. $\Gamma^\tau(\emptyset)$ is stage comparison restricted to computations of length $\leq \tau$.

Definition 2.5. A linear pre-ordering $\precsim$ is a stage comparison if for each $\sigma \in \mathrm{fld}(\precsim)$

$$\precsim \upharpoonright \{\sigma' \mid \sigma' \precsim \sigma\} = \Gamma(\{\sigma' \mid \sigma' < \sigma\}).$$

Lemma 2.6. If $\precsim$ is a stage comparison that is not a pre well-ordering, then the set of true computations form an initial segment of $\precsim$.

proof Let $\precsim_\tau = \Gamma^\tau(\emptyset)$, then by induction $\tau < \kappa$ we show that if σ is in the non-wellfounded part, then $\precsim_\tau$ is an initial segment of

$$\precsim_\sigma = \precsim \upharpoonright \{\sigma' \mid \sigma' \precsim \sigma\}.$$

The empty relation is an initial segment of every relation. Assume that the claim holds for all $\tau' < \tau$ and let σ, σ' both be in the non-wellfunded part s.t. $\sigma' < \sigma$. For all $\tau' < \tau$ we have that $\precsim_{\tau'}$ is an initial segment of $\precsim_{\sigma'}$, but then

$$\precsim_\tau = \Gamma(\bigcup_{\tau'<\tau} \precsim_{\tau'}) \subseteq \Gamma(\precsim_{\sigma'}) \subseteq \precsim_\sigma .$$

A <u>standard stage comparison</u> is a well founded stage comparison. Since $L_\varkappa$ is not Σ_1-admissible we have the Moschovakis Phenomenon, i.e. infinite descending paths in $L_\varkappa$ for computations coded in $L_\varkappa$ which diverge. This means by the previous lemma that we can uniformly prune away non-standard computations, since a witness to divergence lies below its code. In other words we can always compute a <u>standard</u> stage comparison from a stage comparison.

Returning to the set A, let C be the set of B_β's that are stage comparisons. If two of them are incomparable, then we know that $A \notin L_\varkappa$. If they are all comparable, let $\precsim_A$ be the standard part of the union of them. Compute $\|\precsim_A\| = \varkappa'$, then if $\varkappa' = \varkappa$ (which can be decided recursively), then $A \notin L_\varkappa$. If $\|\precsim_A\| < \varkappa$, we can ask:

$$A \in L_{\alpha' + \|\precsim_A\| + \omega + 1} \;?$$

If it is, then $A \in L_\varkappa$. If not there would be a stage comparison extending $\precsim$, but nonetheless of lower order of constructibility than A. But then $\precsim$ would be some B_β, a contradiction. Thus we know that $A \notin L_\varkappa$. This handles $A \subseteq \alpha$, so now consider arbitrary $x \in V$. Proceeding by induction on the rank of x, if $\exists z \in x$ such that we have computed $z \notin L_\varkappa$, then $x \notin L_\varkappa$. Thus assume that we have computed $z \in L_\varkappa$ for all $z \in x$ and compute

$$\sup_{z \in x} O(z) = \varkappa' .$$

If $\kappa' \geq \kappa$ then $x \notin L_\kappa$; otherwise $\kappa' < \kappa$ and w.l.o.g. $x \subseteq \kappa'$. Effectively pass to $f_{\kappa'} : \kappa <\!\!-\!\!> \kappa'$ and apply the above for subsets of α.

Now let $E(\gamma)$ be Σ_1-admissible. The following proposition suffices to show that $E(\gamma) = L_\kappa$ is $\underset{\sim}{RE}$.

<u>Proposition 2.7.</u> If $E(\gamma) = L_\kappa$ for some $\gamma < \kappa$ is Σ_1-admissible, then $\exists x \in L_\kappa$ such that,

$$\kappa_r^x = \kappa .$$

<u>proof</u> see either Sacks [1980] or Sacks-Griffor [1980].

<u>Corollary 2.8.</u> If L_κ is as in the proposition, then L_κ is $\underset{\sim}{RE}$ $(= \Sigma_1)$.

<u>proof</u> the x of the proposition satisfies $\kappa_r^x = \kappa$ and the Sacks characterization of κ_r in this setting yields that $\forall y \in L_\kappa$

$$\kappa_r^{x,y} \geq \kappa_r^x .$$

Thus $(\exists z \in L_\kappa)(\forall y \in L_\kappa)[\kappa_r^{z,y} > O(y)]$ which gives that L_κ is indexical and hence $\underset{\sim}{RE}$.

<u>Remark.</u> The characterization used above for κ_r^x was first proved in the setting of Kleene Recursion in normal functionals of higher type (of which E-recursion is a generalization) by Harrington [1973].

We have completely answered the question of when an E-closure, countable or uncountable, is $\underset{\sim}{RE}$. In the special case of a singular greatest cardinal of uncountable cofinality we have in fact shown that L_κ is $\underset{\sim}{REC}$.

§ 3 Limit E-closed L_κ.

If L_κ is E-closed, countable but not an E-closure we shall determine whether L_κ is $\underset{\sim}{RE}$ with the exception of one case. The uncountable case is complicated by our inability to construct a 'bounded generic' (as we did where L_κ was the E-closure of one of its elements) and our inability to construct a full generic over L_κ, if the cardinality of κ is a singular uncountable cardinal of L. We shall indicate after each result the extension to the uncountable case (if there is one).

The one countable case which is an exception is of some interest. In this case L_κ has a greatest cardinal which has uncountable cofinality in L_κ. In addition, we have that for all τ such that $gc(\kappa) < \tau < \kappa$ we can effectively find the collapse of τ to $gc(\kappa)$. If κ itself is $\underset{\sim}{RE}$, then we can proceed, using the club filter on $cf^{L_\kappa}(gc(\kappa))$, to show that L_κ is $\underset{\sim}{RE}$*). We consider it an interesting open question whether κ is $\underset{\sim}{RE}$ in this case. One can assume that L_κ is the limit of a sequence of E-closed ordinals of type κ itself, for otherwise there is a failure of Σ_1-bounding which will witness that κ is RE: at level τ ask whether all witnesses to this instance of Σ_1-bounding have appeared.

Now suppose that L_κ is E-closed and $\forall x \in L_\kappa [E(x) \in L_\kappa]$. The first case we consider is when L_κ has a greatest cardinal $gc(\kappa)$ and

$$L_\kappa \models \text{"}gc(\kappa) \text{ is regular"}$$

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*) We show here that the order of constructibility function is computable.

Theorem 3.1. If $L_\varkappa$ is countable, E-closed and $(\forall x \in L_\varkappa)$ $[E(x) \in L_\varkappa]$ and

$$L_\varkappa \models \text{'gc}(\varkappa) \text{ exists and is regular'},$$

then $L_\varkappa$ is not $\underset{\sim}{RE}$.

proof suppose $L_\varkappa$ is $\underset{\sim}{RE}$ and let $a \in L_\varkappa$ $e \in \omega$ be such that $\forall x \in V$

$$\{e\}(a,x)\downarrow \iff x \in L_\varkappa. \text{ Let}$$

$O(a) = \gamma < \varkappa$ and consider the following notion of forcing:

$$\mathbb{P} = \{f : gc(\varkappa) \to \{0,1\} \mid \overline{\overline{f}}^{L_\varkappa} < gc(\varkappa)\}.$$

Remark A general remark is in order on what is meant by 'generic' subset of $\mathbb{P}$ over a structure. Unless explicitely stated otherwise we shall write 'bounded generic' for that generic constructed by effective transfinite recursion on the ordinal of the structure over which it is generic. Bounded since only sentences of bounded rank are decided - the reader is directed to Sacks [1980] for details. Two points are worth mentioning:

(i) the ordinal we are building a new subset of must be regular from the point of view of the structure we build the bounded generic with respect to;

(ii) the collection of Gödel numbers for sentences to be decided must be enumerated in an effective way by $gc(\varkappa)$ (or the ordinal we are building a new subset of).

Now let G be $\mathbb{P}$-generic/$_{L_\varkappa}$, then

$$\{e\}(a,G)\uparrow \qquad (*)$$

Case 1 $gc(\varkappa) = \omega$, then let

$$\gamma = O(a) \text{ and consider } E(\gamma) \in L_\varkappa.$$

If $G_o \subseteq \mathbb{P}$ is $\mathbb{P}$-generic$/_{E(\gamma)}$, then $G_o \in L_\varkappa$ and hence

$$E(\gamma)[G_o] \models \{e\}(a,G_o)\downarrow. \text{ By the}$$

forcing lemma $\exists p \in \mathbb{P}$ such that

$$p \Vdash \{e\}(a,\underline{G})\downarrow. \text{ Now if}$$

we take G_1 $\mathbb{P}$-generic$/_{L_\varkappa}$ such that $p \in G_1$, then

$$\{e\}(a,G_1)\downarrow, \text{ which is absurd.}$$

(Note that $\mathbb{P}$ in this case is just the Cohen poset for adding a new real.).

Case 2 $gc(\varkappa) > \omega$ in which case $L_\varkappa$ thinks that $\mathbb{P}$ is $\aleph_1$-closed. Taking G as in (*), then $L_\varkappa[G] \models \{e\}(a,G)\uparrow$ and hence $\exists p \in \mathbb{P}$ s.t.

$$p \Vdash \{e\}(a,\underline{G})\uparrow$$

p is actually an 'encoding' of an infinite descending path. Let

$$\gamma = \max(O(p),O(a))$$

(w.l.o.g. $\gamma > gc(\varkappa)$) and consider $E(\gamma) E L_\varkappa$. $E(\gamma)$ will have either that it is the E-closure of $gc(\varkappa)$ or of γ relative to a parameter. In the second case substitute

$$\mathbb{P}' = \{f : \gamma \to \{0,1\} \mid \overline{\overline{f}}^{E(\gamma)} < \gamma\}$$

and take G_1 $\mathbb{P}$ (or $\mathbb{P}'$) - bounded generic/$E(\gamma)$ s.t. $p \in G_1$. Absoluteness between $L_\varkappa$ and $E(\gamma)$ is maintained since $E(\gamma)$

is an initial segment of $L_\varkappa$ above $gc(\varkappa)$. Thus we have that

$$E(\gamma)[G_1] \models \{e\}(a,G_1)\uparrow \quad \text{contradicting}$$

the choice of e,a.

<u>Remark</u> If $L_\varkappa$ is uncountable, but

$$L \models '\overline{\overline{\varkappa}} \text{ is regular}', \text{ then the}$$

preceeding argument works without change.

We now consider the case where $gc(\varkappa)$ is singular. The following general theorem will be useful.

<u>Theorem 3.2</u> <u>Let</u> $\alpha < \beta$ be ordinals such that $cf(\beta) \leq \alpha$ by some function f recursive in α,β and some $\delta < \alpha$. <u>Then</u> $cf(\beta) \leq \alpha$ by some function recursive in α,β.

<u>proof</u> Let $g : \alpha \to \beta$ be a list of 'computation tuples' over β such that $(\exists\delta < \alpha)[g(\delta)\downarrow]$. The intuition here is that we attempt to carry out a search for $\delta < \alpha$ in question and we either compute it effectively, and hence the witness to $cf(\beta) \leq \alpha$, or we don't and in so doing (not doing) obtain a witness to $cf(\beta) \leq \alpha$. Background to selection in abstract recursion can be found in Harrington-MacQueen [1976]. For the strategy in dynamic proofs of selection see Kirousis [1978] and later Griffor-Normann [1982].

Let

$$\min(g) = \min\{\|g(\delta)\| \mid \delta < \alpha\}.$$

If $E(\beta) \models cf(\beta) > \alpha$, we know that $\min(g)$ is computable by some recursive function $M(g)$. In general it is sufficient for $M(g)$ to be defined that $\min(g)$ exist. If $M(g) < \min(g)$ this means that we have

$$E_{M(g)+1}(\alpha) \models cf(\beta) \leq \alpha .$$

Now let $g(\delta)$ be an index for $\exists f$ recursive in δ,α,β witnessing that $cf(\beta) \leq \alpha$. Since $\min(g)$ exists we have that $M(g)\downarrow$.

If $\min(g) = M(g)$ we have computed the level at which the collapsing map is constructed. If $M(g) < \min(g)$, this is because we know at that ordinal that $cf(\beta) \leq \alpha$.

Thus in both cases we can find from $M(g)$ an f collapsing the cofinality of β below $\alpha+1$.

<u>Corollary 3.3</u> If $\gamma > gc(\varkappa)$, let f_γ be the least (in the sense of $\leq_L$) collapse of γ to $gc(\varkappa)$. if for some $a, \gamma_0 < \varkappa$ we have that

$$(\forall_\gamma > \gamma_0)(\exists z < gc(\varkappa))[f_\gamma \leq_E a, \gamma_0, gc(\varkappa), \gamma, z],$$

<u>then</u> the function $\gamma \mapsto f_\gamma$ is uniformly computable in $\gamma_0, a, gc(\varkappa)$ and a $gc(\varkappa)$ - enumeration of γ_0.

<u>proof</u> We proceed by induction on $\gamma > \gamma_0$. $\gamma = \gamma_0$ is trivial. If $\gamma > \gamma_0$, let α_γ be so large that all $\gamma' < \gamma$ are collapsed to $gc(\varkappa)$. Let $\alpha \geq \alpha\gamma$:

$$\text{if } L_{\alpha_\gamma} \models \overline{\overline{\gamma}} > gc(\varkappa), \text{ then}$$

$$L_\alpha \models \gamma = (gc(\varkappa))^+, \text{ where}$$

τ^+ is the successor cardinal of τ. By the theorem there is an α recursive in γ, a, γ_0, $gc(\varkappa)$ and the collapse of γ_0 such that

$$L_\alpha \models cf(\gamma) \leq gc(\varkappa).$$

But a successor cardinal is regular, so this singularity will demonstrate that $\overline{\overline{\gamma}} = gc(\varkappa)$ and the collapsing map can be computed.

We can now reap the benefit of this interplay between selection and collapsing maps to handle some of the cases of a singular greatest cardinal.

<u>Remark</u> If $\varkappa$ itself is not $\underset{\sim}{\mathrm{RE}}$, then $L_\varkappa$ is not $\underset{\sim}{\mathrm{RE}}$ since we can effectively determine whether a set is an ordinal.

We shall assume in theorems 3.4 and 3.5 that $\varkappa$ is $\underset{\sim}{\mathrm{RE}}$. As previously remarked we regard the question whether $\varkappa$ is $\underset{\sim}{\mathrm{RE}}$ in this situation as an interesting open question. In addition, we introduce:

(*) in some parameter in $L_\varkappa (\exists \alpha \geq gc(\varkappa))\ (\forall \gamma > \alpha)(\exists \tau < gc(\varkappa))[f_\gamma \leq_E \gamma, \tau]$, where

$$f_\gamma : \gamma \longleftrightarrow gc(\varkappa)$$

$L_\varkappa$ here is a limit of E-closures and (*) expresses the fact that $gc(\varkappa)$ is also the greatest cardinal locally.

<u>Theorem 3.4</u> Suppose that $L_\varkappa$ is countable, E-closed but not an E-closure such that $L_\varkappa$ has a greatest cardinal $gc(\varkappa)$:

$$L_\varkappa \models \omega = cf(gc(\varkappa)) < gc(\varkappa), \text{ then}$$

(i) there exists an unbounded ω-sequence through $gc(\varkappa)$ not in $L_\varkappa$, then $L_\varkappa$ is not $\underset{\sim}{\mathrm{RE}}$;

(ii) all ω-sequences through $gc(\varkappa)$ are in $L_\varkappa$, then

(*) $\Rightarrow L_\varkappa$ is $\underset{\sim}{\mathrm{RE}}$ and

$\neg$(*) $\Rightarrow L_\varkappa$ is not $\underset{\sim}{\mathrm{RE}}$ (note that the assumption that $\varkappa$ is $\underset{\sim}{\mathrm{RE}}$ is only used in the case where we show that $L_\varkappa$ <u>is</u> $\underset{\sim}{\mathrm{RE}}$.).

<u>Proof</u> (i) we require a lemma of Sacks (see Sacks-Griffor [1980]):

<u>Lemma 3.5</u> (Sacks) Suppose $L_\varkappa$ is E-closed and not Σ_1-admissible such that

$$L_\varkappa \models \omega = cf(gc(\varkappa)) < gc(\varkappa) \text{ then}$$

$\{x \mid x \subseteq gc(\varkappa) \wedge x \in L_\varkappa\}$ is $\underset{\sim}{RE}$ iff all unbounded ω-sequences through $gc(\varkappa)$ are in $L_\varkappa$.

<u>Remark</u> Sacks' proof is an application of Judy Green's compactness theorem [1974] and a selection result due to Kirousis and Moschovakis. The same proof gives the result in the situation described in the theorem.

Returning to (i) of the theorem: by Sacks' lemma $\{x \mid x \subseteq gc(\varkappa) \wedge x \in L_\varkappa\}$ is not $\underset{\sim}{RE}$ and hence $L_\varkappa$ is not $\underset{\sim}{RE}$ (since any procedure for $L_\varkappa$ would give one for $\{x \subseteq gc(\varkappa) \mid x \in L_\varkappa\}$.

To prove (ii) (*): by lemma 3.4 (a) $\{x \subseteq gc(\varkappa) \mid x \in L_\varkappa\}$ is $\underset{\sim}{RE}$. By (*) and Corollary 3.3 there is a parameter $a \in L_\varkappa$ such that the function $\gamma \mapsto f_\gamma$ is uniformly computable in $\gamma_0, a, gc(\varkappa)$ and a $gc(\varkappa)$ enumeration of γ_0. Since $\varkappa$ is $\underset{\sim}{RE}$ in order to enumerate $L_\varkappa$ proceed as follows: given $x \in V$ assume inductively that we have defined a procedure for all $z \in x$. If that procedure does not converge on all $z \in x$, then diverge. Otherwise we have computed $O(z)$ for all $z \in x$ and let

$$\gamma = \sup_{z \in x} O(z) .$$

If $\gamma \geq \varkappa$ (using $\varkappa$ $\underset{\sim}{RE}$), then diverge. If $\gamma < \varkappa$ we verify this by the procedure given for $\varkappa$. We have a procedure for enumerating $\{x \subseteq gc(\varkappa) \mid x \in L_\varkappa\}$ so pass effectively to f_γ and apply it to $A_\gamma \subseteq gc(\varkappa) \times gc(\varkappa)$ given by

$$A_\gamma = \{\langle\sigma,\tau\rangle \mid f_\gamma^{-1}(\sigma) \leq f_\gamma^{-1}(\tau)\}.$$

By the recursion theorem we have evidently given an enumeration of $L_\varkappa$.

Assume $\neg(*)$ and, toward a contradiction, assume $\exists a \in L_\varkappa$, $\exists e \in \omega$ such that $\forall x \in V$

$x \in L_\varkappa \iff \{e\}(a,x)\downarrow$. We can assume that $a \in OR$ and, by $\neg(*)$, let $\gamma > a$ be least such that $\forall \tau < gc(\varkappa)$ $[f_\gamma \underline{f}_E \gamma, \tau, a]$. Consider $E(\gamma)$ which is an element of $L_\varkappa$.

Remark A straightforward argument shows that

$$E(\gamma) = \{y \mid \varkappa_o^{\gamma,y} < \varkappa_r^{\gamma,\tau} \wedge \tau < gc(\varkappa)\}$$

which by reflection satisfies Σ_1-bounding and is, in fact, Σ_1-admissible. Thus $E(\gamma) = L_\alpha$ Σ_1-admissible and $\alpha^* = \gamma$ (α^* is the Σ_1-projectum of α) and we have

$$f: \alpha \xrightarrow{1-1} \alpha^*, \quad f \in \underset{\sim}{\Sigma}_1(L_\alpha) \quad \text{such}$$

that $\forall \tau < \alpha^*$ $f^{-1}{}''\tau \in L_\alpha$ by Σ_1-bounding. Note that

$L_\alpha \models$ 'γ is the successor cardinal of $gc(\varkappa)$'.

Thus γ is regular of uncountable cofinality in L_α and we consider

$$\mathbb{P} = \{f : \gamma \to \{0,1\} \mid \overline{\overline{f}}^{L_\alpha} < \gamma\} \quad \text{ordered}$$

by inclusion. Then there exists $G_o \subseteq \mathbb{P}$ a $\mathbb{P}$-bounded generic/L_α.

Remark Note that the same effective transfinite recursion using the projection f allows us to build G_o, the difference being that in the case that L_α is not Σ_1-admissible but E-closed, all divergence facts are given by bounded formulae.

Now $G_0 \in L_\kappa$ and by the choice of e,a:

$\{e\}(a,G_0)\downarrow$ and by the genericity of G_0 $L_\alpha[G_0]$ is Σ_1-admissible and

$L_\alpha[G_0] \models \{e\}(a,G_0)\downarrow$. By the forcing lemma $\exists p \in G_0$ such that

$$p \Vdash \{e\}(a,\ )\downarrow .$$

Using the fact that L_κ is countable let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic/$_{L_\kappa}$ such that $p \in G$, then

$$L_\kappa[G] \models \{e\}(a,G)\downarrow \text{ , a contradiction}$$

since $G \notin L_\kappa$. Thus L_κ is not $\underset{\sim}{\mathrm{RE}}$.

We now proceed to the case where the greatest cardinal in L_κ is singular of uncountable cofinality. The principle (*) will play a similar role.

<u>Remark</u> In the uncountable case the positive results will hold. The proofs that L_κ is not $\underset{\sim}{\mathrm{RE}}$ can be carried out if

$\overline{\overline{\kappa}}^L$ is regular, since we need to build generics over L_κ.

<u>Theorem 3.6</u> Suppose that L_κ is countable, E-closed but not an E-closure such that L_κ has a greatest cardinal $gc(\kappa)$:

$$L_\kappa \models \omega < cf(gc(\kappa)) < gc(\kappa) \text{ , } \underline{\text{then}}$$

$(*) \Rightarrow L_\kappa$ is $\underset{\sim}{\mathrm{RE}}$ and

$\neg(*) \Rightarrow L_\kappa$ is not $\underset{\sim}{\mathrm{RE}}$ (i.e. $(*) \Leftrightarrow L_\kappa$ is $\underset{\sim}{\mathrm{RE}}$).

<u>proof</u> Assume (*), then the argument of Theorem 2.1 shows that $O(z)$ is computable on $\{x \mid x \subseteq gc(\kappa) \wedge x \in L_\kappa\}$ using the club filter on $cf(gc(\kappa))$. Since κ itself is $\underset{\sim}{RE}$, we have that $\{x \subseteq gc(\kappa) \mid x \in L_\kappa\}$ is $\underset{\sim}{RE}$ (although indexicality makes no sense in this setting). By (*) and Corollary 3.3. we can proceed as in Theorem 3.4 (ii) (*) to show that L_κ is $\underset{\sim}{RE}$.

The proof that L_κ is not $\underset{\sim}{RE}$ using $\neg$(*) is also as in Theorem 3.4 (ii).

In the uncountable case the positive results will of course hold. The proofs that L_κ is not $\underset{\sim}{RE}$ can be carried out if

$\underset{\kappa}{\neq}^{L}$ is regular, since we need to build

generics over L_κ.

We now consider the case where L_κ has no greatest cardinal (and hence is Σ_1-admissible).

<u>Theorem 3.7</u> If $\kappa > \omega$ is a cardinal of L, then L_κ is E-closed, satisfies MP and

$$L_\kappa \text{ is not } \underset{\sim}{RE} \ (\neq \underset{\sim}{\Sigma}_1(L_\kappa)).$$

<u>proof</u> $\underset{\sim}{RE} \neq \underset{\sim}{\Sigma}_1$ since we have MP
(κ cardinal of $L \Rightarrow (\forall x \in L_\kappa)[E(x) \in L_\kappa]$) and hence the predicates on L_κ:

$$D(e,x) = \{e\}(x)\uparrow \quad \text{are also } \underset{\sim}{\Sigma}_1(L_\kappa).$$

Now suppose for a contradiction that $\exists a \in L_\kappa\ \exists e \in \omega$ such that $\forall x \in V$

$$x \in L_\kappa \iff \{e\}(a,x)\downarrow.$$

It suffices to show that κ itself is not $\underset{\sim}{RE}$ since it is effective to decide whether a set is an ordinal or not. Obviously

$\{e\}(a,\kappa)\uparrow$, thus if we take an elementary substructure of $E(\kappa)$ containing $a \cup \{a\}$ of cardinality less than κ and collapse to L_τ for some $\tau < \kappa$, then $\{e\}(a,\tau)\uparrow$ a contradiction.

If L_κ is countable and has no greatest cardinal we would expect the same result. This is in fact the case.

<u>Theorem 3.8</u> Suppose L_κ is countable and is E-closed with no greatest cardinal. Then L_κ is not $\underset{\sim}{\mathrm{RE}}$.

<u>proof</u> suppose not and let $a \in L_\kappa$ and $e \in \omega$ such that $\forall x \in V$

$$x \in L_\kappa \Longleftrightarrow \{e\}(a,x)\downarrow .$$

W.l.o.g. a is an ordinal so let $\gamma \geq a$ be least regular cardinal in the sense of L_κ. Work over $E(\gamma) \in L_\kappa$ using

$$\mathbb{P} = \{f : \gamma \to \{0,1\} \mid \bar{\bar{f}}^{E(\gamma)} < \gamma\}.$$

$\mathbb{P}$-generics/$E(\gamma)$ can be built in L_κ since

$$L_\kappa \models \text{'}\bar{\bar{E}}(\gamma) \text{ is regular' and}$$

$\mathbb{P}$-generics/L_κ can be built using the countability of L_κ. Proceed now as before to show that $\exists \omega \notin L_\kappa$ s.t.

$$\{e\}(a,\omega)\downarrow \text{, a contradiction.}$$

Thus L_κ is not $\underset{\sim}{\mathrm{RE}}$.

As before this argument can be carried out for uncountable L_κ's as above, if generics over L_κ exist (for example if $\bar{\bar{\kappa}}^L$ is regular.)

<u>Conclusion</u>: The remaining open questions here have to do with certain uncountable situations as indicated. The methods used here rely on the existence of generic objects over uncountable initial segments of L, L_κ, such that $\overline{\overline{\kappa}}^{L}$ is singular. S. Friedman [1980b] has shown that in some cases these generics simply do not exist. We conjecture, however, that the above characterization of which E-closed L_κ's are $\underset{\sim}{\mathrm{RE}}$ holds as well in the uncountable case.

Note also that, with the exception of Theorem 3.4 (a) (ii) (*) we have shown that $O(x)$ (= order for constructibility of x) is computable in the situations where L_κ was shown to be $\underset{\sim}{\mathrm{RE}}$. In addition, the order of constructibility function being computable is absolute. Thus its computability in these situations holds as well for all uncountable L_κ.

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